Math 210C Lecture 2 Notes

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1 Primary Decomposition of Ideals

1.1 Primary decomposition of ideals in noetherian rings

Recall that an ideal I of a commutative ring R is **primary** if whenever $ab \in I$, either $a \in I$ or $b \in \sqrt{I}$. A **primary decomposition** of an ideal I is a collection $\{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ of primary ideals of R with $\bigcap_{i=1}^n \mathfrak{q}_i = I$. It is **minimal** if $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$ and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

Definition 1.1. An ideal is **decomposible** if it has a primary decomposition.

Lemma 1.1. *If I is decomposable, then it has a minimal primary decomposition.*

Definition 1.2. A proper ideal I is **irreducible** if for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $I = \mathfrak{a} \cap \mathfrak{b}$, either $I = \mathfrak{a}$ or $I = \mathfrak{b}$.

Proposition 1.1. *Let R be noetherian. Then every irreducible ideal is primary.*

Proof. Let $I \subseteq R$ be irreudicble. Let $a, b \in I$ with $b \notin I$. For each $n \geq 1$, let $J_n = \{r \in R : a^n r \in I\}$ be an ideal of R. Notice that $J_n \subseteq J_{n+1}$ for all n. By the ascending chain condition, there exists N such that $J_n = J_{n+1}$ for all $n \geq N$.

Let $\mathfrak{a} = (a^N) + I$, $\mathfrak{b} = (b) + I$. We claim that $\mathfrak{a} \cap \mathfrak{b} = I$. Let $c \in \mathfrak{a} \cap \mathfrak{b}$. Then $c = a^n r + q$ with $r \in R$ and $q \in I$ since $c \in \mathfrak{a}$, and $ac \in (ab) + U \subseteq (ab) + I$, since $c \in \mathfrak{b}$. Since $ab \in I$, $ac \in I$. Note that $ac = a^{N+1} r + aq \in I$, so $a^{N+1} r \in I$. So $r \in J_{N+1} = J_N$. Then $a^N r \in I$, so $c \in I$. So $\mathfrak{a} \cap \mathfrak{b} = I$.

Since $b \notin I$, we gmust have $\mathfrak{a} = I$, so $a^N \in I$. That is, $a \in \sqrt{I}$. So I is primary. $\square$

Proposition 1.2. *Let R be noetherian. Then every proper ideal of R is a finite intersection of irreducible ideals.*

Proof. Let X be the collection of proper ideals $I \subsetneq R$ such that I is not a finite intersection of irreducible ideals. We want to show that X is empty. Since R is noetherian, every chain in X has a maximal element. By Zorn's lemma, either $Z = \varnothing$ or X has a maximal element

$\mathfrak{m}$. Since $\mathfrak{m} \in X$, it is not irreducible. So there exist ideals $\mathfrak{a}, \mathfrak{b}$ of R properly containing $\mathfrak{m}$ with $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{m}$. By the maximality of $\mathfrak{m}$ in X, $\mathfrak{a}$ and $\mathfrak{b}$ can be written as finite intersections of irreducible ideals, and thus so can $\mathfrak{m}$. This is a contradiction, so $X = \varnothing$. $\square$

Corollary 1.1 (primary decomposition theorem)**.** *In a noetherian ring, every proper ideal is decomposable.*

1.2 Uniqueness of associated primes

Recall that if $\mathfrak{q} \subsetneq R$ primary, then $\sqrt{\mathfrak{q}} = \mathfrak{p}$ is called the **associated prime** to $\mathfrak{q}$.

Definition 1.3. If $I = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition, then $\sqrt{\mathfrak{q}_i}$ is an **associated prime** of I (relative to this decomposition).

Definition 1.4. An **isolated prime** of I is a minimal element under inclusion in the set of associated primes to I.

Here are examples of primary decompositions.

Example 1.1. Let $(xy^2) \subseteq F[x,y]$. Then $(xy^2) = (x) \cap (y^2)$ is a primary decomposition. The associated primes are (x) and (y) and are isolated.

Example 1.2. Let $I = (xy, y^2) \subseteq F[x,y]$. Then $(xy, y^2) = (x,y)^2 \cap (y)$. We have another decomposition $(xy, y^2) = (x, y^2) \cap (y)$. The associated primes are (a, y) and (y). Here, (y) is isolated.

Lemma 1.2. *Let $I \subseteq R$ be an ideal.*

1. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be prime ideals of R. If $I \subseteq \bigcup_{i=1}^k \mathfrak{p}_i$, then $I \subseteq \mathfrak{p}_i$ for some i.*
2. *Let $\mathfrak{a}_1, \dots, a_k$ be ideals of R. If $\mathfrak{p}$ is prime and $\mathfrak{p} \supseteq \bigcap_{i=1}^k \mathfrak{p}_i$, then $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i.*

Theorem 1.1. *Let I be a decomposable ideal of R. Then the set of associated primes to a minimal primary decomposition of I is independent of the decomposition.*

Proof. Let $\{q_1, \dots, q_k\}$ be a primary decomposition of I. For $a \in R$, let $I_a = \{r \in R : ra \in I\}$ be an ideal of R. Then $I_a = \bigcup_{i=1}^k (\mathfrak{q}_i)_a$. Let $\mathfrak{p}_i - \sqrt{\mathfrak{q}_i}$ for all i. Then $(\mathfrak{q}_i)_a = R$ if $a \in \mathfrak{q}_i$ and $\sqrt{(\mathfrak{q}_i)_a} = \mathfrak{p}_i$ if $a \notin \mathfrak{q}_i$ (exercise using $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary). So $\sqrt{I_a} = \bigcap \sqrt{(\mathfrak{q}_i)_a} = \bigcap_{\substack{i=1 \\ a \notin \mathfrak{q}_i}}^k \mathfrak{p}_i$. For any i, we may choose $a \in \bigcap_{j \neq i} q_j$ with $a \notin \mathfrak{q}_i$ (by the minimality of $\{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$). So $\sqrt{I_a} = \mathfrak{p}_i$. For any $a \in R$ such that $\sqrt{I}$ is prime, by the lemma, $\sqrt{I_a} \supseteq \mathfrak{p}_i$ for some I. So $\sqrt{I_a} = \mathfrak{p}_i$. Thus, the $\mathfrak{p}_i$ are uniquely determined by I. $\square$